Hyperkähler geometry of a cubic fourfold via moduli spaces

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Setting

A cubic fourfold Y is a smooth cubic hypersurface in $\mathbb{P}^5_{\mathbb{C}}$.
Semiorthogonal decomposition (Kuznetsov):

$$\mathrm{D}^b(Y) = \langle \mathrm{Ku}(Y), \mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2)\rangle$$

$\mathrm{Ku}(Y)$ is a **K3 category**.

Aim

Describe the hyperkähler manifolds associated to moduli spaces M_d of rational curves of degree d on Y, as (desingularizations of) moduli spaces of Bridgeland (semi)stable objects in $\mathrm{Ku}(Y)$.

Motivations

1) Explanation of the existence of the symplectic form using derived categories.
2) Birational models via wall-crossing.

Key ingredient: in [1], they construct Bridgeland stability conditions on $\mathrm{Ku}(Y)$. We denote such a stability condition by $\bar{\sigma}$.

Properties of $\mathrm{Ku}(Y)$

- The **Serre functor** of $\mathrm{Ku}(Y)$ is the shift by 2 as for the derived category of a K3 surface:
$$\mathrm{Hom}(A, B[i]) \cong \mathrm{Hom}(B, A[2-i])^* \, \forall A, B \in \mathrm{Ku}(Y).$$
- (Addington, Thomas) The **Mukai lattice** of $\mathrm{Ku}(Y)$ is $\tilde{H}(\mathrm{Ku}(Y), \mathbb{Z}) :=$
$\{\kappa \in K_{\mathrm{top}}(Y) : \chi([\mathcal{O}_Y(i)], \kappa) = 0 \text{ for every } i = 0, 1, 2\}$
with the weight 2 Hodge structure defined by
$\tilde{H}^{2,0}(\mathrm{Ku}(Y)) := v^{-1}(H^{3,1}(Y))$
$\tilde{H}^{1,1}(\mathrm{Ku}(Y)) := v^{-1}(\oplus_p H^{p,p}(Y))$
where $v : K_{\mathrm{top}}(\mathrm{Ku}(Y)) \to \oplus_i H^i(Y, \mathbb{Z})(i)$.
- There exist algebraic classes λ_1, λ_2 in $\tilde{H}(\mathrm{Ku}(Y), \mathbb{Z})$ spanning an A_2-lattice.

Degree 1: Lines

The Fano variety $M_1 := F_Y$ of lines in Y is a smooth projective hyperkähler fourfold deformation equivalent to the Hilbert square on a K3 surface (Beauville, Donagi).

Theorem 1

The Fano variety of lines on Y is a moduli space of stable objects in $\mathrm{Ku}(Y)$ with respect to the Bridgeland stability condition $\bar{\sigma}$, with Mukai vector $\lambda_1+\lambda_2$.

Objects: (Macrì, Stellari) Consider the ideal sheaf $\mathcal{I}_\ell$ of a line $\ell \subset Y$.

$$\rightsquigarrow \mathcal{O}_Y(-1)[1] \to P_\ell \to \mathcal{I}_\ell \quad \text{where } P_\ell \in \mathrm{Ku}(Y).$$

Degree 2: Conics

Assume Y does not contain a plane. Conic curves in Y are residual to lines. $\rightsquigarrow$ $M_2 \to F_Y$ has 3-dimensional fibers.

Degree 3: Twisted cubics

Let Y be a cubic fourfold not containing a plane.

$$s : M_3 \to \mathbb{G}(\mathbb{P}^3, \mathbb{P}^5), \quad C \longmapsto \langle C\rangle \cong \mathbb{P}^3$$
$$s^{-1}(\mathbb{P}^3) = \mathrm{Hilb}^{gtc}(S) \longmapsto \mathbb{P}^3$$

where $S = Y \cap \mathbb{P}^3$ is an irreducible reduced cubic surface.
Geometric picture:(Lehn, Lehn, Sorger, van Straten)
1) The morphism above factorizes through a $\mathbb{P}^2$-fibration $M_3 \to M'_Y$, where M'_Y is a smooth and projective variety of dimension eight.
2) The locus of non CM curves in M'_Y is a Cartier divisor D which can be contracted and the resulting variety M_Y is a smooth projective hyperkähler eightfold.

$$\begin{array}{ccccc} M_3 & \xrightarrow{\mathbb{P}^2\text{-fibr.}} & M'_Y & \hookleftarrow & D \cong \mathbb{P}(T_Y) \\ \downarrow & \swarrow & \downarrow \text{contr.} & & \downarrow \\ \mathbb{G}(\mathbb{P}^3, \mathbb{P}^5) & & M_Y & \hookleftarrow & Y \end{array}$$

M_Y is equivalent by deformation to K3[4] (Addington, Lehn).

Theorem 2

Assume that Y does not contain a plane. Then the LLSvS eightfold M_Y is a moduli space of stable objects in $\mathrm{Ku}(Y)$ with respect to the Bridgeland stability condition $\bar{\sigma}$, with Mukai vector $2\lambda_1 + \lambda_2$.

Objects:(Lahoz, Lehn, Macrì, Stellari) Consider the ideal sheaf $\mathcal{I}_{C/S}$ of a twisted cubic curve C in the cubic surface $S \subset Y$.

$$\rightsquigarrow F_C := \ker(H^0(Y, \mathcal{I}_{C/S}(2)) \otimes \mathcal{O}_Y \xrightarrow{ev} \mathcal{I}_{C/S}(2)).$$

Fact: If C is aCM, then $F_C \in \mathrm{Ku}(Y)$, while in the non CM case $F_C \notin \mathrm{Ku}(Y)$.

$$\rightsquigarrow F'_C := \mathbb{R}_{\mathcal{O}_Y(-1)}(F_C) \in \mathrm{Ku}(Y).$$

Now you would expect **rational quartic curves**. By residuality in a rational cubic scroll this is equivalent to consider:

Elliptic quintics

Objects: Consider the ideal sheaf $\mathcal{I}_{\Gamma/Y}$ of an elliptic quintic curve $\Gamma \subset Y$.

$$\mathrm{D}^b(Y) = \langle \mathcal{O}_Y(-2), \mathcal{O}_Y(-1), \mathrm{Ku}(Y), \mathcal{O}_Y\rangle.$$

Consider

$$P_\Gamma := \mathbb{R}_{\mathcal{O}_Y(-1)}\mathbb{R}_{\mathcal{O}_Y(-2)}\mathbb{L}_{\mathcal{O}_Y}\mathcal{I}_{\Gamma/Y}(1) \in \mathrm{Ku}(Y).$$

- $v(P_\Gamma) = 2\lambda_1 + 2\lambda_2$;
- If $\langle\Gamma\rangle \cong \mathbb{P}^4$, Γ is locally complete intersection and $h^0(\mathcal{O}_\Gamma) = 1$, then consider the cubic threefold
$$X := \langle\Gamma\rangle \cap Y$$
and
$$0 \to \mathcal{O}_X(-1) \to E_\Gamma \to \mathcal{I}_{\Gamma/X}(1) \to 0.$$
Remark: E_Γ has been defined by Markushevich and Tikhomirov in relation with the intermediate Jacobian of a cubic threefold.
Property:
$$P_\Gamma \cong E_\Gamma.$$
- If $\langle\Gamma\rangle \cong \mathbb{P}^3$, Γ is reduced and $h^0(\mathcal{O}_\Gamma) = 1$, then consider the cubic surface $Z := \langle\Gamma\rangle \cap Y$.
$$\Gamma \equiv H \cap Z + \ell_1 + \ell_2 \text{ in } Z.$$
Property:
$$P_\Gamma \cong P_{\ell_1} \oplus P_{\ell_2}.$$

Theorem 3

Assume that Y is generic. Then:
1) E_Γ is $\bar{\sigma}$-stable.
2) Let M be the moduli space of $\bar{\sigma}$-semistable objects in $\mathrm{Ku}(Y)$ with Mukai vector $2\lambda_1 + 2\lambda_2$. Then
$$\mathrm{Sing}(M) \cong \mathrm{Sym}^2 F_Y$$
and M has a symplectic resolution $\tilde{M}$, which is a smooth projective hyperkähler tenfold deformation equivalent to the example constructed by O'Grady.

Intermediate Jacobian

The twisted relative intermediate Jacobian parametrizes 1-cycles of degree 1 in the smooth hyperplane sections of Y. Voisin proved it has a flat projective compactification $\tilde{J}^T$ over $(\mathbb{P}^5)^*$.

$$\tilde{M} \dashrightarrow \tilde{J}^T, E_\Gamma \mapsto c_2(E_\Gamma).$$

Question: Are $\tilde{M}$ and $\tilde{J}^T$ isomorphic for Y very general?

References

[1] A. Bayer, M. Lahoz, E. Macrì, P. Stellari, *Stability conditions on Kuznetsov components*, (Appendix joint also with X. Zhao), arXiv:1703.10839.

[2] C. Li, L. Pertusi, X. Zhao, *Twisted cubics on cubic fourfolds and stability conditions*, arXiv:1802.01134.

[3] C. Li, L. Pertusi, X. Zhao, *Elliptic quintics on cubic fourfolds and O'Grady spaces*, in preparation.